Finite Gauge Theory on Fuzzy S^2 and $\mathbb{C}P^2$

Harold Steinacker

Vienna, November 2004

hep-th/0407089 (H. Grosse, H. S.)

hep-th/0307075, Nucl.Phys. B679 (2004) (H. S.)

Motivation:

- Space-time continuum is an idealization,
  probably *quantized* $\Rightarrow$ "NonCommutative spaces"
  (string theory, quantum gravity)
- explore NCFT: formulation and properties
  UV regularization ?
  new approaches / tools for field theory?
- new physics beyond the NC scale? Simplification?

Strategy:

use nice class of NC spaces to avoid mathematical complications

$\Rightarrow$ "fuzzy" spaces: (J. Madore)

- $\mathcal{M}_N$ ... NC space with finitely many degrees of freedom: N $(\approx \Lambda)$
- $\mathcal{M}_N \to \mathcal{M}$ (manifold) as $N \to \infty$.
- (NC) geometrical structures, symmetries, ...
- typically matrix algebras

$\approx$ NC lattice field theory

Examples:

fuzzy S^2_N, $\mathbb{C}P^2_N$, $S^2_N \times S^2_N$, ...

any (co)adjoint orbit of a Lie group G can be quantized in terms of a finite matrix algebra.

Goal: Study (quantum) field theory on $\mathcal{M}_N$

is regularized (cp. lattice field theory)

can take various scaling limits: e.g.

$$\mathbb{C}P^2_N \quad \to \quad \begin{cases} \mathbb{C}P^2 & \text{(commutative limit)} \\ \mathbb{R}^4_\theta & \text{(quantum plane)} \\ \mathbb{R}^4 & \end{cases}$$

as $N \to \infty$.

Field theory on $\mathbb{C}P^2_N$ is regularized, but

existence of limit $N \to \infty$ is nontrivial

(UV/IR mixing, renormalization)

Peculiarities for fuzzy spaces:

- coadjoint orbits $\Rightarrow$ can use group theory!!
- occur in string theory:
  - D-branes of group manifolds
  - solutions of IKKT matrix model, ...
- Gauge theory on fuzzy spaces can be formulated as (multi)-matrix model
  - well-defined
  - simple: connections, bundles, topology arises automatically
  - useful? computable?

Noncommutative gauge theory in a nutshell

Let $\mathcal{A}$ ... NC algebra of functions on a space

assume: there exist elements $\lambda_i \in \mathcal{A}$ such that

$$\partial_i f(x) = [\lambda_i, f(x)], \qquad f(x) \in \mathcal{A}$$

assume $$[\lambda_i, \lambda_j] = \theta_{ij}(\lambda)$$

consider "covariant coordinates" (one-forms, ...)

$$\boxed{B_i = \lambda_i + A_i}, \qquad A = A_i\text{... gauge field}$$

then:

$$\begin{aligned}[B_i, B_j] - \theta_{ij}(B_i) &= [\lambda_i, A_j] - [\lambda_j, A_i] + [A_i, A_j] \\ &= \quad F_{ij} \quad \text{... field strength}\end{aligned}$$

gauge transformations: $B_i \to U^{-1} B_i U, \quad A_i \to U^{-1}\partial_i U + U^{-1} A U$

Yang-Mills action on $\mathcal{A}$:

$$\boxed{S \quad = Tr(F_{ij}F^{ij}) = Tr(([B_i, B_j] - \theta_{ij}(B))([B^i, B^j] - \theta^{ij}(B)))}$$

obviously gauge invariant under

$$B_i \to U^{-1} B_i U$$

$U \in U(\mathcal{A}) \qquad\qquad \Rightarrow U(1)$ gauge theory

$\mathcal{A} \to \mathcal{A} \otimes Mat(n) \Rightarrow U(n)$ gauge theory

simpler than on commutative spaces: don't need connection

... essentially Matrix Models: $\quad B_i \in \mathcal{A}$ resp. $Gl(\mathcal{H})$.

<u>trouble</u>: usually $\dim(\mathcal{A}) = \infty \qquad$ (e.g. for $\mathbb{R}^4_\theta$)

<u>however</u>: on "fuzzy spaces", $\dim(\mathcal{A}) = finite$.

The fuzzy sphere S^2_N:

(Madore; Grosse, Klimcik, Presnajder ...)

algebra of functions on S^2_N: $\mathcal{A} = Mat(N, \mathbb{C})$

rotations: let λ_i ... N-dim. rep. of $su(2)$: $f \in \mathcal{A} \to J_i \triangleright f := [\lambda_i, f]$

space of functions decomposes into

$$
\begin{aligned}
f \in Mat(N, \mathbb{C}) \cong (N) \otimes (N) \;\; &= (1) \;\oplus\; (3) \;\oplus ... \oplus\; (2N-1) \\
&= (\hat{Y}^0) + (\hat{Y}^1_m) + ... + (\hat{Y}^{N-1}_m)
\end{aligned}
$$

hence: $\exists$ map

$$
\begin{aligned}
S^2_N \;\; &\to \;\; S^2, \\
\hat{Y}^l_m \;\; &\to \;\; Y^l_m
\end{aligned}
$$

coordinates $x_i = \frac{R}{N}\lambda_i$ satisfy

$$
[x_i, x_j] = \frac{iR}{N}\, \varepsilon_{ijk}\, x_k, \qquad x_1^2 + x_2^2 + x_3^2 = R^2.
$$

Gauge theory on S^2_N as Matrix Model

(Grosse, Klimcik, Watamuras, H.S. ...)

consider "covariant coordinates"

$$B_i = \lambda_i + A_i \qquad \in Fun(S^2_N) = Mat(N, \mathbb{C})$$

since $[\lambda_j, \lambda_k] = i\varepsilon_{ijk}\lambda_i$, define field strength:

$$\boxed{\begin{aligned} F_{jk} &= [B_j, B_k] - i\varepsilon_{ijk}B_i \\ &= [\lambda_j, A_k] - [\lambda_k, A_j] - i\varepsilon_{ijk}A_i \\ &\cong dA + AA \end{aligned}}$$

try action:

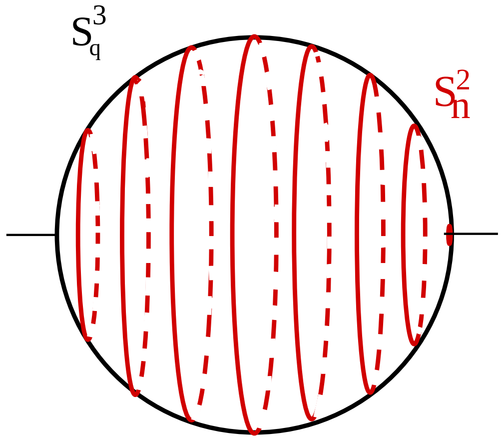
$$\boxed{\begin{aligned} S &= \tfrac{1}{g^2}\int([B_j, B_k] - i\varepsilon_{ijk}B_i)([B^j, B^k] - i\varepsilon^{ijk}B^i) \\ &= \tfrac{1}{g^2}\int F_{jk}F^{jk} \end{aligned}}$$

Solutions:

$$F_{jk} = 0 \quad \Leftrightarrow \quad [B_j, B_k] = i\varepsilon_{ijk} B_i$$

any rep. of $su(2)$ is a solution!

interpretation as "multi-brane configurations" in string theory



geometry is dynamical !

To obtain gauge theory on S^2_N, additional term necessary:

$$S_{YM} = \frac{1}{g^2} \int (B_i B^i - \lambda_i \lambda^i)^2 + F_{jk} F^{jk}$$

prefers solution $B_i = \lambda_i$, suppresses other (reducible) solutions

... YM on fuzzy sphere, 2 tangential gauge degrees of freedom

(Multi-) Matrix model!

generalization to $U(k)$ Yang-Mills: same action, $B_i \in Mat(kN, \mathbb{C})$

---

note: NC gauge theory: action determines geometry,

gauge fields $A = B - \lambda$ ... fluctuations of NC geometry

Quantization

$$\boxed{Z[J] = \int dB_i \, e^{-S_{YM}(B_i) \ + B_i J^i},} \quad B_i \in Mat(N, \mathbb{C})$$

- well-defined (finite!)
- invariant under gauge-trafos and $SO(3)$ rotation

---

known: Z can be computed for $U(k)$ YM on S^2 (Migdal, Rusakov)

can also do that on S^2_N (hep-th/0307075), using methods from random matrix theory (Itzykson-Zuber formula)

Hence:

YM on fuzzy sphere $\rightarrow$ YM on classical sphere, is calculable

Fuzzy $\mathbb{C}P^2_N$

(Grosse & Strohmaier, Balachandran etal, ...)

Imagine $\mathbb{C}P^2 \subset su(3) \cong \mathbb{R}^8$

Fuzzy version:

$$\mathbb{C}P^2_N = End(V_N, \mathbb{C}),$$

V_N ... irrep of $su(3)$ with highest weight $N\Lambda_2$.

$$\xi_a = T_a|_{V_N}, \quad x_a = \Lambda_N\, \xi_a, \qquad \Lambda_N \approx \frac{R}{N}$$

and satisfy the relations

$$\begin{aligned} [x_a, x_b] &= \tfrac{i}{2}\Lambda_N f_{abc}\, x_c \\ g^{ab} x_a x_b &= R^2, \\ d_c^{ab} x_a x_b &= R\, \tfrac{2N/3+1}{\sqrt{\frac{1}{3}N^2+N}}\, x_c. \end{aligned}$$

Gauge theory on fuzzy $\mathbb{C}P^2_N$

(H. Grosse & H.S.)

8 covariant coordinates $\Rightarrow$ (4 gauge fields, 4 auxiliary fields)

$$B_a = \xi_a + A_a$$

Yang-Mills action: multi-matrix model

$$\boxed{S = \frac{1}{g} \int F_{ab}F_{ab} + \frac{1}{N} D_a D_a}$$

where

$$\begin{aligned} F_{ab} &= i[B_a, B_b] + \tfrac{1}{2} f_{abc} B_c \\ D_a &= d_{abc} B_b B_c - (\tfrac{2N}{3} + 1) B_a \end{aligned}$$

note: additional term

$$\frac{1}{N} D_a D_a = 4N(A_1^2 + A_2^2 + A_3^2 + \frac{1}{9} A_8^2)$$

gives large mass to the transversal fields, 4 gauge fields $A_{4,5,6,7}$ survive

Monopole solutions: Exact solution of e.o.m:

$$C_a = \begin{pmatrix} \alpha\xi_a^{(N-m)} & 0 \\ 0 & 0 \end{pmatrix} = \xi_a + A_i\sigma^i$$

$\Rightarrow A_a$ finite, explicit monopole field (except at "sphere at infinity")

Field strength is finite,

$$F = -m\,\eta, \qquad \eta\text{... symplectic form}$$

$$c_1 = m \qquad \text{... } 1^{st} \text{ Chern number}$$

"Instanton" solutions: similar exact solution of equation of motion,

first and second Chern number $c_1, c_2 \neq 0$

nontrivial rank 2 bundle: "instanton + monopole" (not self-dual)

Quantization

$$\boxed{Z[J] = \int dB_a \, e^{-S(B_a) \; + B_a J^a},} \qquad B_a \in Mat(D_N, \mathbb{C})$$

nice properties:

- well-defined (finite!)
- invariant under gauge-trafos and $SU(3)$ rotation

Conclusions:

- gauge theory on fuzzy spaces = (multi-) Matrix Model
- provides nonperturbative definition for finite N
- "smart" lattice gauge theory, preserving symmetries
- $\exists$ scaling limit $S^2_N \to \mathbb{R}^2_\theta$, $\mathbb{C}P^2_N \to \mathbb{R}^4_\theta$, ...

Questions/Problems:

- scaling limit $N \to \infty$? properties?
- calculate observables, develop tools